XXZ Bethe states as highest weight vectors of the sl_2 loop algebra at roots of unity

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Abstract

We prove some part of the conjecture that regular Bethe ansatz eigenvectors of the XXZ spin chain at roots of unity are highest weight vectors of the sl_2 loop algebra. Here q is related to the XXZ anisotropic coupling Δ by $\Delta = (q + q^{-1})/2$, and it is given by a root of unity, $q^{2N} = 1$, for a positive integer N. We show that regular XXZ Bethe states are annihilated by the generators $\bar{x}_k^+$'s, for any N. We discuss, for some particular cases of $N = 2$, that regular XXZ Bethe states are eigenvectors of the generators of the Cartan subalgebra, $\bar{h}_k$'s. Here the loop algebra $U(L(sl_2))$ is generated by $\bar{x}_k^{\pm}$ and $\bar{h}_k$ for $k \in \mathbf{Z}$, which are the classical analogues of the Drinfeld generators of the quantum loop algebra $U_q(L(sl_2))$. A representation of $U(L(sl_2))$ is called highest weight if it is generated by a vector Ω which is annihilated by the generators $\bar{x}_k^+$'s and such that Ω is an eigenvector of the $\bar{h}_k$'s. We also discuss the classical analogue of the Drinfeld polynomial which characterizes the irreducible finite-dimensional highest weight representation of $U(L(sl_2))$.

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I. INTRODUCTION

The XXZ spin chain is one of the most important exactly solvable quantum systems[1,2]. The Hamiltonian under the periodic boundary conditions is given by

$$H_{XXZ} = -J \sum_{j=1}^{L} \left(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z \right) . \tag{1.1}$$

Here the parameter Δ expresses the anisotropic coupling between adjacent spins, and it is related to the q parameter of the quantum group $U_q(sl_2)$ by the relation $\Delta = (q + q^{-1})/2$.

Recently it was shown that when q is a root of unity, the Hamiltonian commutes with the generators of the sl_2 loop algebra[3]. The symmetry of the XXZ Hamiltonian is enhanced at the root of unity. The spectra of the XXZ spin chain have been studied numerically[4,5], and some generalizations of the sl_2 loop algebra symmetry have also been derived for some trigonometric vertex models[6]. Furthermore, the spectral degeneracy associated with the sl_2 loop algebra has been shown for the eight-vertex model and some elliptic IRF models[7,8]. However, it has been a conjecture that the Bethe ansatz eigenvectors are of highest weight with respect to the sl_2 loop algebra.

In this paper we show several rigorous results supporting the highest-weight conjecture. Based on the algebraic Bethe ansatz method, we show some part of the conjecture that a regular Bethe ansatz eigenvector of the XXZ spin chain should give a highest weight vector of the sl_2 loop algebra in the sense of the Drinfeld realization of the quantum loop algebra $U_q(L(sl_2))$. Here the regular Bethe ansatz eigenvector is defined by such an eigenvector that is constructed from the Bethe ansatz wave-function where all the rapidities are finite.

Let us consider the classical analogue of the Drinfeld realization of $U_q(L(sl_2))$[9,10]. The sl_2 loop algebra $U(L(sl_2))$ is generated by the classical analogues of the Drinfeld generators, $\bar{x}_k^{\pm}$ and $\bar{h}_k$ for $k \in \mathbf{Z}$. In the sense of the Drinfeld realization, a representation of the loop algebra $U(L(sl_2))$ is highest weight if it is generated by a vector Ω which is annihilated by the generator $\bar{x}_k$ for any integer k and such that Ω is a common eigenvector of generators of the Cartan subalgebra, $\bar{h}_k$'s. For the case of any N, we show that a regular XXZ Bethe state $|R\rangle$ at the root of unity is annihilated by the generators $\bar{x}_k^+$'s: $\bar{x}_k^+|R\rangle = 0$, for all $k \in \mathbf{Z}$. For some particular case of $N = 2$, we show that the regular XXZ Bethe state $|R\rangle$ is a common eigenvector of the generators $\bar{h}_k$'s of the Cartan subalgebra. Thus, the proof of the highest-weight conjecture is partially completed for the case of $N = 2$. For the vacuum state, we introduce a method for calculating the eigenvalues of the Cartan generators $\bar{h}_k$'s. Some version of the highest-weight conjecture has been addressed in Ref.[3] from a different viewpoint. It is also addressed in Ref.[5].

We also discuss a method for calculating the classical analogue of the Drinfeld polynomial. Through some examples of the Drinfeld polynomial, we suggest a conjecture that the irreducible representation of $U(L(sl_2))$ generated by a regular XXZ Bethe state with R down spins on L sites should be given by $2^{(L-2R)/N}$, when L and R are integral multiples of N. The result is consistent with the analytic computation of the degeneracy shown for the eight-vertex model and some elliptic SOS models associated with the eight-vertex model[7,8]. In association with the sl_2 loop algebra, the Drinfeld polynomial is also addressed in Ref.[5] with a different method.

II. SL_2 LOOP ALGEBRA AND THE DRINFELD REALIZATION

A. Loop algebra symmetry of the XXZ spin chain

We review some results on the sl_2 loop algebra symmetry of the XXZ spin chain. We first introduce the quantum group $U_q(sl_2)$. The generators $S^{\pm}$ and S^Z satisfy the following relations

$$[S^+, S^-] = \frac{q^{2S^Z} - q^{-2S^Z}}{q - q^{-1}}, \quad [S^Z, S^{\pm}] = \pm S^{\pm} \tag{2.1}$$

Here the parameter q is generic. The comultiplication is given by

$$\Delta(S^{\pm}) = S^{\pm} \otimes q^{-S^Z} + q^{S^Z} \otimes S^{\pm}, \quad \Delta(S^Z) = S^Z \otimes I + I \otimes S^Z \tag{2.2}$$

We consider the Lth tensor product of spin 1/2 representation $V^{\otimes L}$. The representations of the generators $S^{\pm}$ and S^Z are given by

$$\begin{aligned} q^{S^Z} &= q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2} \\ S^{\pm} &= \sum_{j=1}^{L} q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2} \otimes \sigma_j^{\pm} \otimes q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2} \end{aligned} \tag{2.3}$$

Considering the automorphism of the $U_q(L(sl_2))$, we may introduce the following operator

$$T^{\pm} = \sum_{j=1}^{L} q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2} \otimes \sigma_j^{\pm} \otimes q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2} \tag{2.4}$$

Let us now define the operators $S^{\pm(N)}$ and $T^{\pm(N)}$ by

$$S^{\pm(N)} = \left(S^{\pm}\right)^N / [N]!, \quad T^{\pm(N)} = \left(T^{\pm}\right)^N / [N]! \tag{2.5}$$

Here we have used the q-integer $[n] = (q^n - q^{-n})/(q - q^{-1})$ and the q-factorial $[n]! = [n][n-1] \cdots [1]$. We also note that q is not a root of unity. Then, we can show the following expressions

$$\begin{aligned} S^{\pm(N)} = \sum_{1 \le j_1 < \cdots < j_N \le L} & q^{\frac{N}{2}\sigma^Z} \otimes \cdots \otimes q^{\frac{N}{2}\sigma^Z} \otimes \sigma_{j_1}^{\pm} \otimes q^{\frac{(N-2)}{2}\sigma^Z} \otimes \cdots \otimes q^{\frac{(N-2)}{2}\sigma^Z} \\ & \otimes \sigma_{j_2}^{\pm} \otimes q^{\frac{(N-4)}{2}\sigma^Z} \otimes \cdots \otimes \sigma_{j_N}^{\pm} \otimes q^{-\frac{N}{2}\sigma^Z} \otimes \cdots \otimes q^{-\frac{N}{2}\sigma^Z} \quad . \end{aligned} \tag{2.6}$$

For $T^{\pm(N)}$ we have

$$\begin{aligned} T^{\pm(N)} = \sum_{1 \le j_1 < \cdots < j_N \le L} & q^{-\frac{N}{2}\sigma^Z} \otimes \cdots \otimes q^{-\frac{N}{2}\sigma^Z} \otimes \sigma_{j_1}^{\pm} \otimes q^{-\frac{(N-2)}{2}\sigma^Z} \otimes \cdots \otimes q^{-\frac{(N-2)}{2}\sigma^Z} \\ & \otimes \sigma_{j_2}^{\pm} \otimes q^{-\frac{(N-4)}{2}\sigma^Z} \otimes \cdots \otimes \sigma_{j_N}^{\pm} \otimes q^{\frac{N}{2}\sigma^Z} \otimes \cdots \otimes q^{\frac{N}{2}\sigma^Z} \quad . \end{aligned} \tag{2.7}$$

Here we recall that q is generic.

Let the symbol $\tau_{6V}(v)$ denotes the (inhomogeneous) transfer matrix of the six-vertex model. We now take the parameter q a root of unity. We consider the limit of sending q to a root of unity: $q^{2N} = 1$. Then we can show the (anti) commutation relations[3] in the sector of $S^Z \equiv 0$ (mod N)

$$S^{\pm(N)}\tau_{6V}(v) = q^N \tau_{6V}(v) S^{\pm(N)}, \qquad T^{\pm(N)}\tau_{6V}(v) = q^N \tau_{6V}(v) T^{\pm(N)} \tag{2.8}$$

Here we recall that S^Z denotes the Z-component of the total spin operator.

Since the XXZ Hamiltonian H_{XXZ} is given by the logarithmic derivative of the (homogeneous) transfer matrix $T_{6V}(v)$, we have in the sector $S^Z \equiv 0$ (mod N)

$$[S^{\pm(N)}, H_{XXZ}] = [T^{\pm(N)}, H_{XXZ}] = 0. \tag{2.9}$$

Thus, the operators $S^{\pm(N)}$ and $T^{\pm(N)}$ commute with the XXZ Hamiltonian in the sector $S^Z \equiv 0$ (mod N) .

Let us now consider the algebra generated by the operators $S^{\pm(N)}$ and $T^{\pm(N)}$[3]. When q is a primitive $2N$th root of unity, or a primitive Nth root of unity with N odd, we consider the identification in the following[3]:

$$E_0^+ = S^{+(N)}, \quad E_0^- = S^{-(N)}, \quad E_1^+ = T^{-(N)}, \quad E_1^- = T^{+(N)}, \quad H_0 = -H_1 = \frac{2}{N} S^Z \, . \tag{2.10}$$

It is shown in Ref.[3] that the operators $E_j^{\pm}, H_j$ for $j = 0, 1$, satisfy the defining relations of the algebra $U(L(sl_2))$.

Noticing the automorphism

$$\theta(E_0^{\pm}) = E_1^{\pm}\, , \quad \theta(H_0) = H_1 \tag{2.11}$$

we may take the identification in the following:

$$E_0^+ = T^{-(N)}, \quad E_0^- = T^{+(N)}, \quad E_1^+ = S^{+(N)}, \quad E_1^- = S^{-(N)}, \quad H_0 = -H_1 = -\frac{2}{N} S^Z \, . \tag{2.12}$$

B. Classical analogue of the Drinfeld realization

Let us review briefly the Drinfeld realization of the quantum loop algebra[9,10]. The quantum affine algebra $U_q(\hat{sl_2})$ is isomorphic to the associative algebra over $\mathbf{C}$ with generators $x_k^{\pm}$ ($k \in \mathbf{Z}$), h_k ($k \in \mathbf{Z} \setminus \{0\}$), $K^{\pm 1}$, central element $C^{\pm 1}$, and the following relations:

$$\begin{aligned}
CC^{-1} = C^{-1}C = KK^{-1} &= K^{-1}K = 1 \\
[h_k, h_\ell] &= \delta_{k,-\ell} \frac{1}{k} [2k] \frac{C^k - C^{-k}}{q - q^{-1}} \\
Kh_k &= h_k K, K x_k^{\pm} K^{-1} = q^{\pm 2} x_k^{\pm} \\
[h_k, x_\ell^{\pm}] &= \pm \frac{1}{k} [2k] C^{\mp(k+|k|)/2} x_{k+\ell}^{\pm} \\
x_{k+1}^{\pm} x_k^{\pm} - q^{\pm 2} x_\ell^{\pm} x_{k+1}^{\pm} &= q^{\pm 2} x_k^{\pm} x_{\ell+1}^{\pm} - x_{\ell+1}^{\pm} x_k^{\pm}
\end{aligned}$$

$$
\begin{aligned}
[x_k^+, x_\ell^-] &= \frac{1}{q-q^{-1}}\left(C^{k-\ell}\psi_{k+\ell} - \phi_{k+\ell}\right) \\
\sum_{k=0}^{\infty} @\psi_k u^k &= K \exp\left((q-q^{-1})\sum_{k=1}^{\infty} h_k u^k\right) \\
\sum_{k=0}^{\infty} @\phi_{-k} u^{-k} &= K^{-1} \exp\left(-(q-q^{-1})\sum_{k=1}^{\infty} h_{-k} u^{-k}\right)
\end{aligned}
\tag{2.13}
$$

Let us consider the classical analogue of the Drinfeld realization[10]. Putting $q = \exp\epsilon$, we take the limit of ϵ to zero: $q = 1 + \epsilon + \cdots$. Hereafter we assume the trivial center: $C^{\pm 1} = 1$. We define the classical analogs of the Drinfeld generators as follows.

$$
\begin{aligned}
K &= q^{\bar{h}_0} \\
h_k &= \bar{h}_K + O(\epsilon) \qquad (k \in \mathbf{Z} \setminus \{0\}) \\
x_k^{\pm} &= \bar{x}_k^{\pm} + O(\epsilon) \qquad (k \in \mathbf{Z})
\end{aligned}
\tag{2.14}
$$

From the last two relations of the set of defining relations of eqs. (2.13), we have the following relations

$$
\begin{aligned}
\psi_k &= 2\epsilon \bar{h}_k + O(\epsilon^2) \qquad (k \geq 1) \\
\phi_{-k} &= -2\epsilon \bar{h}_{-k} + O(\epsilon^2) \qquad (k \geq 1).
\end{aligned}
\tag{2.15}
$$

Then, from the classical limit of the quantum loop algebra, we have the following relations:

$$
\begin{aligned}
[\bar{h}_k, \bar{h}_\ell] &= 0 \\
[\bar{h}_k, x_\ell^{\pm}] &= \pm 2 x_{k+\ell}^{\pm} \\
\bar{x}_{k+1}^{\pm}\bar{x}_\ell^{\pm} - \bar{x}_\ell^{\pm}\bar{x}_{k+1}^{\pm} &= \bar{x}_k^{\pm}\bar{x}_{\ell+1}^{\pm} - \bar{x}_{\ell+1}^{\pm}\bar{x}_k^{\pm} \\
[\bar{x}_k^+, \bar{x}_\ell^-] &= \bar{h}_{k+\ell}
\end{aligned}
\tag{2.16}
$$

Here $k, \ell \in \mathbf{Z}$.

Let us consider the classical limit of the isomorphism of the Drinfeld realization of the quantum loop group to the quantum affine algebra $U_q(\hat{sl_2})$. It is the isomorphism between the sl_2 loop algebra and the classical limit of the quantum affine algebra given by the following:

$$
\begin{aligned}
& E_1^{\pm} \mapsto \bar{x}_0^{\pm} \\
& E_0^{+} \mapsto \bar{x}_1^{-} \qquad E_0^{-} \mapsto \bar{x}_{-1}^{+} \\
& -H_0 = H_1 \mapsto \bar{h}_0
\end{aligned}
\tag{2.17}
$$

Applying the isomorphism to the identification (2.12), we have the correspondence:

$$
\bar{x}_0^+ = T^{-(N)}, \quad \bar{x}_0^- = T^{+(N)}, \quad \bar{x}_1^+ = S^{+(N)}, \quad \bar{x}_{-1}^- = S^{-(N)}, \quad \bar{h}_0 = \frac{2}{N} S^Z \tag{2.18}
$$

Let us now give the definition of a highest weight representation for the loop algebra $U(L(sl_2))$. They are given by the following:

A representation of the sl_2 loop algebra is highest weight if it is generated by a vector Ω which is annihilated by the generator $\bar{x}_k^+$ for all $k \in \mathbf{Z}$ and such that Ω is an eigenvector of the Cartan generator $\bar{h}_k$ for $k \in \mathbf{Z}$.

Thus, a vector Ω is highest weight if it is annihilated by $\bar{x}_k^+$ for all integer k, and it is an eigenvector of the generator $\bar{h}_k$ for all integer k.

III. FORMULAS OF ALGEBRAIC BETHE ANSATZ

A. R matrix and L operator

Let us summarize some formulas of the algebraic Bethe ansatz[11–13]. We define the R matrix of the XXZ spin chain by

$$R(z-w) = \begin{pmatrix} f(w-z) & 0 & 0 & 0 \\ 0 & g(w-z) & 1 & 0 \\ 0 & 1 & g(w-z) & 0 \\ 0 & 0 & 0 & f(w-z) \end{pmatrix} \tag{3.1}$$

where $f(z-w)$ and $g(z-w)$ are given by

$$f(z-w) = \frac{\sinh(z-w-2\eta)}{\sinh(z-w)} \quad g(z-w) = \frac{\sinh(-2\eta)}{\sinh(z-w)} \tag{3.2}$$

Hereafter we shall write $f(t_1 - t_2)$ by f_{12} for short.

We now introduce L operators for the XXZ spin chain

$$L_n(z) = \begin{pmatrix} L_n(z)_1^1 & L_n(z)_2^1 \\ L_n(z)_1^2 & L_n(z)_2^2 \end{pmatrix} = \begin{pmatrix} \sinh(zI_n + \eta\sigma_n^z) & \sinh 2\eta\, \sigma_n^- \\ \sinh 2\eta\, \sigma_n^+ & \sinh(zI_n - \eta\sigma_n^z) \end{pmatrix} \tag{3.3}$$

Here I_n and σ_n^a $(n = 1, \ldots, L)$ are acting on the n th vector space V_n. The L operator is an operator-valued matrix which acts on the auxiliary vector space V_0. The symbols $\sigma^{\pm}$ denote $\sigma^+ = E_{12}$ and $\sigma^- = E_{21}$, and $\sigma^x, \sigma^y, \sigma^z$ are the Pauli matrices

In terms of the R matrix and L operators, the Yang-Baxter equation is expressed as

$$R(z-t)\,(L_n(z) \otimes L_n(t)) = (L_n(t) \otimes L_n(z))\, R(z-t) \tag{3.4}$$

We define the monodromy matrix T by the product: $T(z) = L_L(z) \cdots L_2(z) L_1(z)$, and the transfer matrix $\tau_{6V}(z)$ by the trace: $\tau_{6V}(z) = \mathrm{Tr}\, T(z)$. The matrix elements of $T(z)$

$$T(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \tag{3.5}$$

satisfy the commutation relations derived from the Yang-Baxter equations such as $B_1 B_2 = B_2 B_1$ and

$$A_1 B_2 = f_{12} B_2 A_1 - g_{12} B_1 A_2\,, \quad D_1 B_2 = f_{21} B_2 D_1 - g_{21} B_1 D_2 \tag{3.6}$$

Here we recall B_1 denotes $B(t_1)$. Furthermore we can show[13]

$$
\begin{aligned}
C_0 B_1 \cdots B_n = {} & B_1 B_2 \cdots B_n C_0 \\
& + \sum_{j=1}^{n} B_1 \cdots B_{j-1} B_{j+1} \cdots B_n \, g_{0j} \left\{ A_0 D_j \prod_{k \neq j} f_{0k} f_{kj} - A_j D_0 \prod_{k \neq j} f_{k0} f_{jk} \right\} \\
& - \sum_{1 \leq j < k \leq n} B_0 B_1 \cdots B_{j-1} B_{j+1} \cdots B_{k-1} B_{k+1} \cdots B_n \\
& \quad \times g_{0j} g_{0k} \{ A_j D_k f_{jk} \prod_{\ell \neq j,k} f_{j\ell} f_{\ell k} + A_k D_j f_{kj} \prod_{\ell \neq j,k} f_{k\ell} f_{\ell j} \}
\end{aligned}
\tag{3.7}
$$

Let us denote by $|0\rangle$ the vector where all the spins are up. Then we can show

$$
A(z)|0\rangle = a(z)|0\rangle \,, \quad D(z)|0\rangle = d(z)|0\rangle \tag{3.8}
$$

where $a(z) = \sinh^L(z + \eta)$ and $d(z) = \sinh^L(z - \eta)$.

Making use of the commutation relations (3.6) one can show that the vector $B_1 B_2 \cdots B_R |0\rangle$ is an eigenvector of the XXZ spin Hamiltonian if the rapidities $t_1, t_2, \ldots, t_R$ satisfy the Bethe ansatz equations

$$
\frac{a_j}{d_j} = \prod_{k \neq j} \left(\frac{f_{kj}}{f_{jk}} \right) \quad \text{for} \quad j = 1, \cdots R \,. \tag{3.9}
$$

B. Formulas with infinite rapidities

Let us normalize the operators $A(z)$'s as follows

$$
\begin{aligned}
\hat{A}(z) &= A(z)/n(z) \qquad \hat{B}(z) = B(z)/(g(z)n(z)) \\
\hat{D}(z) &= D(z)/n(z) \qquad \hat{C}(z) = B(z)/(g(z)n(z)) \,.
\end{aligned}
\tag{3.10}
$$

Here the normalization factor $n(z)$ is given by $n(z) = \sinh^L z$. Then, we can show the following:

$$
\begin{aligned}
\hat{A}(\pm\infty) &= q^{\pm S^Z} \qquad \hat{B}(\infty) = -T^- \quad \hat{B}(-\infty) = -S^- \\
\hat{D}(\pm\infty) &= q^{\mp S^Z} \qquad \hat{C}(\infty) = -S^+ \quad \hat{C}(-\infty) = -T^+
\end{aligned}
\tag{3.11}
$$

Here, we recall that $S^{\pm}$ and S^Z are defined on the Lth tensor product of spin 1/2 representations. Hereafter, we shall write $\hat{C}(\infty)$'s simply as $\hat{C}_\infty$'s.

We now derive quite a useful formula. Sending z_0 to infinity in eq. (3.7), we have the following:

$$
\begin{aligned}
\hat{C}_\infty \, B_1 \cdots B_M \; 0\rangle = {} & \sum_{j=1}^{M} B_1 \cdots B_{j-1} B_{j+1} \cdots B_M |0\rangle \\
& \times e^{t_j} \left(q^{(L/2)-(M-1)} d_j \prod_{k \neq j} f_{kj} - q^{-(L/2)+(M-1)} a_j \prod_{k \neq j} f_{jk} \right)
\end{aligned}
\tag{3.12}
$$

where $t_1, \ldots, t_M$ are given arbitrary. Applying the formula (3.12) N times, we have an important formula in the following:

$$
\begin{aligned}
&\left(\hat{C}_\infty\right)^N B_1\cdots B_M|0\rangle \\
&= \sum_{S_N\subseteq\Sigma_M}\sum_{P\in\mathcal{S}_N}\left(\prod_{j\in\Sigma_M\setminus S_N} B_j\right)|0\rangle\ \exp(\sum_{j\in S_N} t_j) \\
&\prod_{n=1}^{N}\left(q^{(L/2)-(M-n)} d_{j_{Pn}}\prod_{k\neq j_{P1},\cdots,j_{Pn}} f_{kj_{Pn}} - q^{-(L/2)+(M-n)} a_{j_{Pn}}\prod_{k\neq j_{P1},\cdots,j_{Pn}} f_{j_{Pn}k}\right)
\end{aligned}
\tag{3.13}
$$

Here the set Σ_M is given by $\Sigma_M=\{1,2,\cdots,M\}$ and S_N denote the subset of Σ_M with N elements. The symbol $\mathcal{S}_N$ denotes the symmetry group of N elements. The symbol $P\in\mathcal{S}_N$ means that P is a permutation of N letters. Here we recall that $t_1,\ldots,t_M$ are given arbitrary.

IV. PROOF OF THE ANNIHILATION OF REGULAR XXZ BETHE VECTORS BY THE DRINFELD GENERATORS

A. The Nth power of C operators acting on regular Bethe states

Let us now discuss the proof for the annihilation property: when $t_1,\ldots,t_R$ are finite and they satisfy the Bethe ansatz equations (3.9), then we have

$$
S^{+(N)}\, B_1\cdots B_R|0\rangle = 0 \tag{4.1}
$$

We may assume $R>N$, otherwise it is trivial. Making use of the Bethe ansatz equations (3.9) we have the following formula:

$$
\begin{aligned}
&\prod_{n=1}^{N}\left(q^{(L/2)-(R-n)} d_{j_{Pn}}\prod_{k\neq j_{P1},\cdots,j_{Pn}} f_{kj_{Pn}} - q^{-(L/2)+(R-n)} a_{j_{Pn}}\prod_{k\neq j_{P1},\cdots,j_{Pn}} f_{j_{Pn}k}\right) \\
&= \left(\prod_{\ell=1}^{N} a_{j_{P\ell}}\right)\prod_{n=1}^{N}\left(\prod_{k\notin S_N} f_{j_{Pn}k}\right)\prod_{n=1}^{N}\left(q^{(L/2)-(R-n)}\prod_{k\neq j_{P1},\cdots,j_{Pn}} f_{j_{Pn}j_{P\ell}} - q^{-(L/2)+(R-n)}\prod_{\ell=1}^{n-1} f_{j_{P\ell}j_{Pn}}\right)
\end{aligned}
\tag{4.2}
$$

By induction on N, we can show the following formula:

$$
\sum_{P\in\mathcal{S}_N}\prod_{n=1}^{N}\left(x_n\prod_{\ell=1}^{n-1} f_{j_{Pn}j_{P\ell}} - y_n\prod_{\ell=1}^{n-1} f_{j_{P\ell}j_{Pn}}\right) = \prod_{n=1}^{N}(x_n-y_n)\times\sum_{P\in\mathcal{S}_N}\prod_{n=1}^{N}\prod_{\ell=1}^{n-1} f_{j_{P\ell}j_{Pn}} \tag{4.3}
$$

Applying the formula (4.3) we obtain

$$
\begin{aligned}
&\left(\hat{C}_\infty\right)^N B_1\cdots B_R|0\rangle \\
&= \sum_{S_N\subseteq\Sigma_R}\left(\prod_{\ell\in\Sigma_R\setminus S_N} B_\ell\right)|0\rangle\ \exp(\sum_{j\in S_N} t_j)\left(\prod_{j\in S_N} a_j\right)\prod_{j\in S_N}\left(\prod_{k\notin S_N} f_{jk}\right) \\
&\prod_{n=1}^{N}\left(q^{(L/2)-(R-n)} - q^{-(L/2)+R-n}\right)\sum_{P\in\mathcal{S}_N}\prod_{n=1}^{N}\prod_{\ell=1}^{n-1} f_{j_{P\ell}j_{Pn}}
\end{aligned}
\tag{4.4}
$$

Noting the relation

$$\hat{C}(\infty)^N = (-1)^N \left(S^+\right)^N \, , \tag{4.5}$$

we divide the both hand sides of (4.4) by the q factorial $[N]!$, and we have for the case of generic q

$$\begin{aligned} & S^{+(N)} \, B_1 B_2 \cdots B_R|0\rangle \\ & = (-1)^N \sum_{S_N \subseteq \Sigma_R} \sum_{P \in \mathcal{S}_N} \left(\prod_{\ell \in \Sigma_R \setminus S_N} B_\ell \right) |0\rangle \, \exp(\sum_{j \in S_N} t_j) \left(\prod_{j \in S_N} a_j \right) \prod_{j \in S_N} \left(\prod_{k \notin S_N} f_{jk} \right) \\ & \quad \times \left[\begin{array}{c} \frac{L}{2} - R + N \\ N \end{array} \right]_q (q - q^{-1})^N \left(\sum_{P \in \mathcal{S}_N} \prod_{n=1}^{N} \prod_{\ell=1}^{n-1} f_{j_{P\ell} j_{Pn}} \right) \end{aligned} \tag{4.6}$$

Here, we have defined the q-binomial by

$$\left[\begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]!}{[m-n]! \, [n]!} \quad , \text{ for } \quad m \geq n \, . \tag{4.7}$$

B. The case of roots of unity

Let us now consider the case when q is a root of unity. When q is a $2N$th root of unity or an Nth root of unity with N odd, we can show the following equality

$$\sum_{P \in \mathcal{S}_N} \prod_{n=1}^{N} \prod_{\ell=1}^{n-1} f_{j_{P\ell} j_{Pn}} = 0 \tag{4.8}$$

Let us define a function $F(z_1, \cdots, z_N)$ by

$$F(z_1, \cdots, z_N) = \sum_{P \in \mathcal{S}_N} \prod_{n=1}^{N} \prod_{\ell=1}^{n-1} f_{P\ell, Pn} \tag{4.9}$$

Then we can show that there is no pole for $F(z_1, \cdots, z_N)$ with respect to any variable z_j, and also that $F(z_1, \cdots, z_N)$ vanishes when one of the variables is sent to infinity: $z_j = \infty$ for some j. Therefore we have the equality (4.8). It follows from (4.6) and (4.8) that the action of the operator $S^{+(N)}$ on the regular Bethe ansatz state $B_1 \cdots B_R|0\rangle$ is given by zero.

By a similar method with $S^{+(N)}$, we can show that the operator $T^{+(N)}$ makes the regular Bethe ansatz eigenstate vanishes. Thus, we have obtained

$$S^{+(N)} \, B_1 \cdots B_M|0\rangle = 0 \, , \quad T^{+(N)} \, B_1 \cdots B_M|0\rangle = 0. \tag{4.10}$$

V. EIGENVALUES OF THE CARTAN GENERATORS ON REGULAR BETHE STATES

We discuss a method by which we can calculate the eigenvalue of the generator $\bar{h}_k$ on a regular Bethe state. For an illustration, we consider the case of $N = 2$.

Let us calculate the eigenvalue of the operator $S^{+(2)}T^{-(2)}$ on a regular Bethe state with R down-spins: $|R\rangle = B_1 \cdots B_R|0\rangle$. Here we recall that R denotes the number of regular rapidities. We set $M = R+2$. We assume that $t_1, \ldots, t_R$ satisfy the Bethe ansatz equations, while z_{R+1} and z_{R+2} are sent to infinity, later. We now consider the formula (**??**) for the case of $N = 2$ in the following:

$$\left(\hat{C}_\infty\right)^2 B_1 B_2 \cdots B_M|0\rangle = \sum_{S_2 \subset \Sigma_M} \sum_{P \in \mathcal{S}_2} \prod_{\ell \in \Sigma_M \setminus S_2} B_\ell|0\rangle \, \exp(\sum_{j \in S_2}) \times \prod_{n=1}^{2} \left(a_{j_{Pn}} q^{-L/2} \prod_{k \neq j_{P1}, \ldots, j_{Pn}} (q f_{j_{Pn}\,k}) - d_{j_{Pn}} q^{L/2} \prod_{k \neq j_{P1}, \ldots, j_{Pn}} (q^{-1} f_{k\,j_{Pn}}) \right) \quad (5.1)$$

Here we recall $T^- = -\hat{B}(\infty)$ and $S^+ = -\hat{C}(\infty)$. In eq. (5.1) we have four cases for the set S_2: $S_2 = \{R+1, R+2\}$, $\{j, R+2\}$, $\{j, R+1\}$ and $\{j_1, j_2\}$. Here, $j, j_1, j_2 \in \{1, 2, \ldots, R\}$. For an illustration, we highlight the term with $S_2 = \{R+1, R+2\}$ in the following:

$$\begin{aligned} &\left(\hat{C}_\infty\right)^2 \prod_{\ell \in \Sigma_M} B_\ell|0\rangle \\ &= \prod_{\ell=1}^{R} B_\ell|0\rangle \, e^{z_{R+1}+z_{R+2}} \sum_{P \in \mathcal{S}_2} \prod_{n=1}^{2} \left(a_{j_{Pn}} q^{-L/2} \prod_{k \neq j_{P1}, \ldots, j_{Pn}} (q f_{j_{Pn}\,k}) - d_{j_{Pn}} q^{L/2} \prod_{k \neq j_{P1}, \ldots, j_{Pn}} (q^{-1} f_{k\,j_{Pn}}) \right) \\ &+ \cdots . \end{aligned} \quad (5.2)$$

After sending z_{R+1} to infinity, we take the limit of sending z_{R+2} to infinity. Then, we have the following:

$$\begin{aligned} &\lim_{z_{R+2}\to\infty} \left(\lim_{z_{R+1}\to\infty} \left(\hat{C}_\infty\right)^2 B_1 \cdots B_R \hat{B}_{R+1} \hat{B}_{R+2}|0\rangle \right) \\ &= \prod_{\ell=1}^{R} B_\ell|0\rangle \, [2]^2 \left\{ \left({}_L C_2 + [3] \sum_{k=1}^{R} e^{4t_k} \right) - [2] L \sum_{k=1}^{R} e^{2t_k} + [2]^2 \sum_{j<k} e^{2(t_j+t_K)} \right\} \\ &+ \sum_{j=1}^{R} \left[\hat{B}_\infty \prod_{\ell \neq j} B_\ell|0\rangle \left(-(q^2+q^{-2}) \sum_{k \neq j} e^{2t_k} + (L-1) - e^{2t_j} \right) - \frac{\delta \hat{B}_\infty}{\delta \epsilon_{R+2}} \prod_{\ell \neq j} B_\ell|0\rangle \right] \\ &\times [\frac{L}{2} - R - 2][2]^2 (q - q^{-1}) e^{t_j} a_j \prod_{k \neq j} f_{jk} \\ &+ \sum_{1 \le j_1 < j_2 \le R} \left(\hat{B}_\infty\right)^2 \prod_{\ell=1; \ell \neq j_1, j_2}^{R} B_\ell|0\rangle \, e^{t_{j_1}+t_{j_2}} (q - q^{-1})^2 \\ &\times [\frac{L}{2} - R - 2][\frac{L}{2} - R - 3][2] \, a_{j_1} a_{j_2} \prod_{k \neq j_1, j_2} f_{j_1\,k} f_{j_2\,k} @ \end{aligned} \quad (5.3)$$

Here, $\epsilon_{R+2} = \exp(-2z_{R+2})$. We note that q is generic, so far.

We multiply the normalization factor $[2]^2$ with (5.3). Then we take the limit of sending q to a root of unity. If $L/2 - R \equiv 0 \pmod 2$, then all the unwanted terms vanish. We have

$$S^{+(2)} T^{-(2)} \, B_1 \cdots B_R |0\rangle = \left({}_L C_2 + [3] \sum_{k=1}^{R} e^{4t_k} \right) B_1 \cdots B_R |0\rangle \tag{5.4}$$

VI. CLASSICAL ANALOGUE OF THE DRINFELD POLYNOMIAL

A. Definition

Let us denote the eigenvalue of the generator $\bar{h}_k$ on Ω as

$$\bar{h}_k \Omega = \bar{d}_k^{+} \Omega \qquad - \bar{h}_{-k} \Omega = \bar{d}_{-k}^{-} \Omega \tag{6.1}$$

for $k \geq 1$ and $k \in \mathbf{Z}$.

The classical analogue of the Drinfeld polynomial $P(u)$ satisfies

$$\sum_{k=1}^{\infty} \bar{d}_k u^k = -u \frac{P^{'}(u)}{P(u)} \, , \quad \sum_{k=1}^{\infty} \bar{d}_{-k} u^{-k} = degP - u \frac{P^{'}(u)}{P(u)} \, . \tag{6.2}$$

B. Some useful formulas

Recall that $\bar{x}_k^{\pm}$ and $\bar{h}_k$ denote the classical analogues of the Drinfeld generators $x_k^{\pm}$ and h_k, respectively. Then, we can show

$$\bar{x}_k^{+} = \frac{1}{2^k} \left(ad_{\bar{h}_1} \right)^k \bar{x}_0^{+} \, , \quad \bar{x}_{-k}^{+} = \frac{1}{2^k} \left(ad_{\bar{h}_{-1}} \right)^k \bar{x}_0^{+} \tag{6.3}$$

for $k > 0$. Here $\bar{h}_1$ is given by

$$\bar{h}_1 = [\bar{x}_0^{+}, \bar{x}_1^{-}] \, , \quad \bar{h}_{-1} = (-1)[\bar{x}_{-1}^{+}, \bar{x}_0^{+}] \, , \tag{6.4}$$

From the second identification (2.12) we have

$$\bar{h}_1 = [S^{+(N)}, T^{-(N)}] \, , \quad \bar{h}_{-1} = [S^{-(N)}, T^{+(N)}] \, , \tag{6.5}$$

In the second identification (2.12), we have

$$\bar{x}_0^{+} = S^{+(N)} \tag{6.6}$$

Thus, we can express the generators $\bar{x}_k^{+}$ in terms of $S^{\pm(N)}$ and $T^{\pm(N)}$ for any integer k.

The equation (6.3) can also be formulated as

$$[\bar{h}_1, \bar{x}_k] = 2\bar{x}_{k+1}^{+} \tag{6.7}$$

Making use of eqs. (6.7) and (4.10), we can recursively show that the generator $\bar{x}_k^{+}$ vanishes on regular XXZ Bethe ansatz eigenvectors at roots of unity. Here, we have assumed that

the action of $\bar{h}_1$ on a regular Bethe ansatz state can be calculated with the method of §5. For $N = 2$, it is finished as shown in §5. Precisely speaking, however, it is a conjecture for $N > 2$.

In order to calculate the evaluation parameters of the Drinfeld polynomial, we can use the following:

$$\bar{h}_{k+1} = \frac{1}{2}[[\bar{h}_k, \bar{x}_0^+], \bar{x}_1^-] \tag{6.8}$$

C. An example of the classical analogue of the Drinfeld polynomial

For $L = 6$ and $N = 3$, the Drinfeld polynomial of the degenerate eigenspace for $|0\rangle$ is given by

$$P(u) = (1 - a_1 u)(1 - a_2 u) \tag{6.9}$$

where a_1 and a_2 are given by $10 \pm 3\sqrt{11}$. The roots are distinct and the degree of P is two, so that the dimension is given by $2^2 = 4$.

VII. DISCUSSION

Through the classical analogue of the Drinfeld polynomial, we can calculate the dimension of the multiplets of the sl_2 loop algebra.

The dimensions of the sl_2 multiplets should be consistent with that of the XYZ model: $2^{(L-2M)/N}$[7,8]. In fact, we can show that when $L = 2M$ and the rapidities $t_1 \ldots, t_M$ are finite-valued solutions to the Bethe ansatz equations we have

$$S^{\pm(N)} B_1 \cdots B_M |0\rangle = T^{\pm(N)} B_1 \cdots B_M |0\rangle = 0 \tag{7.1}$$

More details of computation of the classical analogues of the Drinfeld polynomials should be reported elsewhere.

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